

HARMONIC MAP METHODS FOR WILLMORE SURFACES

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ABSTRACT. In this note we demonstrate how the analogy between the harmonic Gauss map of a constant mean curvature surface and the harmonic conformal Gauss map of a Willmore surface can be used to obtain results on Willmore surfaces.

1. INTRODUCTION

For an immersion $f : M \rightarrow \mathbb{R}^3$ of a Riemann surface into Euclidean 3-space the harmonicity of the Gauss map $N : M \rightarrow S^2$ characterises by the Ruh–Vilms theorem [RV70] a constant mean curvature surface. In particular, methods from harmonic map theory can be used to study constant mean curvature surfaces, e.g. [EW83, Hit90, Bob91, PS89]. Viewing Euclidean 3-space as the imaginary quaternions equipped with the inner product $\langle a, b \rangle = -\operatorname{Re}(ab)$, $a, b \in \operatorname{Im} \mathbb{H}$, the Gauss map of a constant mean curvature surface can be seen as a complex structure on the trivial \mathbb{H} -bundle over M .

For Willmore surfaces $f : M \rightarrow \mathbb{R}^4$ a similar characterisation is known [Eji88, Rig87]: a conformal immersion is Willmore if and only if its conformal Gauss map is harmonic. Since the Willmore property is a conformal invariant, we study Willmore surfaces up to Möbius transformations, i.e., we consider conformal immersions $f : M \rightarrow S^4$ from a Riemann surface into the 4-sphere. Our main tools are from Quaternionic Holomorphic Geometry [PP98, BFL⁺02, FLPP01], in particular, we view the quaternionic projective line $\mathbb{HP}^1 = S^4$ as the 4-sphere and use the general linear group $\operatorname{GL}(2, \mathbb{H})$ to study the conformal geometry of the 4-sphere. In this setup, the conformal Gauss map can be seen as a complex structure on the trivial \mathbb{H}^2 -bundle over M , and in this sense, Willmore surfaces can be considered as a generalisation of a rank 1 theory to a rank 2 problem. The aim of this paper is to demonstrate in two examples how this analogy can be exploited to obtain results for Willmore surfaces.

2. WILLMORE SURFACES IN THE 4-SPHERE

We first recall some basic facts on Willmore surfaces, for details compare [BFL⁺02]. In this paper we consider conformal immersions $f : M \rightarrow S^4$ from a Riemann surface M into S^4 where we identify the 4-sphere $S^4 = \mathbb{HP}^1$ with the quaternionic projective line. Since a point in \mathbb{HP}^1 gives a line in \mathbb{H}^2 , a map $f : M \rightarrow \mathbb{HP}^1$ can be identified with a line subbundle $L \subset \underline{\mathbb{H}}^2$ of the trivial bundle $\underline{\mathbb{H}}^2 = M \times \mathbb{H}^2$ by $L_p = f(p)$. The differential of f is under this identification given by

$$\delta : \Gamma(L) \rightarrow \Omega^1(\underline{\mathbb{H}}^2/L), \psi \mapsto \pi d\psi$$

where d is the trivial connection on $\underline{\mathbb{H}}^2$ and $\pi : \underline{\mathbb{H}}^2 \rightarrow \underline{\mathbb{H}}^2/L$ is the canonical projection. An immersion is *conformal* if there exists a complex structure $S \in \Gamma(\operatorname{End}(\underline{\mathbb{H}}^2))$, that is a

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smooth map into the quaternionic endomorphism with $S^2 = -1$, which stabilises L and is compatible with the complex structure J_M on the tangent space TM , that is,

$$(1) \quad * \delta = S \delta = \delta S$$

where $*\delta(X) = \delta(J_M X)$ for $X \in TM$. Note that since $SL \subset L$, the complex structure S induces a complex structure on \mathbb{H}^2/L which we denote again by S . The condition (1) does not determine the complex structure S uniquely; to fix S we consider the conformal Gauss map of f : we decompose the differential $dS = (dS)' + (dS)''$ of a complex structure S into $(1, 0)$ and $(0, 1)$ parts

$$(dS)' = \frac{1}{2}(dS - S * dS), \quad (dS)'' = \frac{1}{2}(dS + S * dS)$$

with respect to the complex structure S . Denoting by

$$(2) \quad A = \frac{1}{2}(*dS)', \quad Q = -\frac{1}{2}(*dS)'',$$

the *Hopf fields* of S , the derivative of S can be written as $dS = 2(*Q - *A)$. Note that the Hopf fields anti-commute with S and thus satisfy

$$(3) \quad *A = SA = -AS, \quad \text{and} \quad *Q = -SQ = QS.$$

Definition 1. Let $f : M \rightarrow S^4$ be a conformal immersion from a Riemann surface M into the 4-sphere. The *conformal Gauss map* of f is a complex structure $S \in \Gamma(\text{End}(\mathbb{H}^2))$, $S^2 = -1$, such that $*\delta = S\delta = \delta S$ and

$$(4) \quad \text{im } A \subset L \quad \text{or, equivalently,} \quad L \subset \ker Q.$$

A complex structure $S \in \text{End}(\mathbb{H}^2)$ can be identified with a 2-sphere $S' \subset S^4$ by $S' = \{l \in \mathbb{HP}^1 \mid Sl = l\}$. This way, a complex structure $S \in \Gamma(\text{End}(\mathbb{H}^2))$ gives a sphere congruence, and the condition $SL \subset L$ says that $f(p) \in S'(p)$ is a point on the sphere given by S at p . Moreover, the conformality equation (1) says that S envelopes f that is, $f(p) \in S'(p)$, and the tangent space of $S'(p)$ at $f(p)$ and $d_p f(T_p M)$ coincide in an oriented way. Finally, the condition (4) shows that the mean curvature vectors of $f(M)$ and of $S'(p)$ coincide at $f(p)$. In other words, the conformal Gauss map is the *mean curvature sphere congruence* of the conformal immersion f .

Definition 2. A conformal immersion $f : M \rightarrow S^4$ of a compact Riemann surface M into the 4-sphere is called *Willmore surface* if f is a critical point of the *Willmore energy*

$$W(f) = 2 \int_M \langle A \wedge *A \rangle$$

under compactly supported variations of f (where the conformal structure of M may change under the variation). Here A is the Hopf field of the conformal Gauss map S of f , and $\langle B \rangle = \text{tr } B$ is the real trace of an endomorphism $B \in \text{End}(\mathbb{H}^2)$.

Remark 3. The *Willmore energy* of a conformal immersion $f : M \rightarrow S^4$ is given by

$$W(f) = \int_M (|\mathcal{H}|^2 - K - K^\perp) \text{vol}_{f^*h}$$

where the mean curvature vector \mathcal{H} , the Gaussian curvature K , and the normal bundle curvature K^\perp are computed with respect to a conformally flat metric h on S^4 .

Since the energy functional of the conformal Gauss map coincides with the Willmore energy of f up to topological constants, one can show:

Theorem 4 ([Eji88, Rig87], for the quaternionic formulation [BFL⁺02]). *A conformal immersion $f : M \rightarrow S^4$ is Willmore if and only if the conformal Gauss map S of f is harmonic, that is, if and only if*

$$(5) \quad d * A = 0 \quad \text{or, equivalently,} \quad d * Q = 0.$$

3. WILLMORE SEQUENCES

In the case of harmonic maps $N : M \rightarrow \mathbb{CP}^n$ of a Riemann surface into complex projective space the $(0,1)$ and the $(1,0)$ part of the derivative of N give a sequence of harmonic maps [GS80],[DZ80]. This sequence can be used to prove the Eells–Wood theorem [EW83] and its generalisations [BJRW88, Wol88, Uhl89]: a harmonic map from a genus g Riemann surface into the complex projective line \mathbb{CP}^1 is holomorphic or anti-holomorphic if the degree of the harmonic map is bigger than $g-1$. For example, a constant mean curvature sphere has a holomorphic unit normal map and thus is a round sphere as first observed by Hopf [Hop83]. On the other hand, immersed constant mean curvature tori have unit normal maps of degree zero and the harmonic sequence does not terminate. This case leads to the theory of spectral curves and the construction of constant mean curvature tori from algebraically completely integrable systems [PS89, Hit90, Bob91].

We explain a corresponding $\bar{\partial}$ and ∂ construction on the harmonic conformal Gauss map to obtain Willmore sequences, [LP08]. Using the Willmore sequence we get a unified proof of the classification results for Willmore spheres [Bry84, Mon00, Eji88], and Willmore tori with non-trivial normal bundle [LPP05]. In particular, it only remains to study integrable system methods and spectral curves for Willmore tori with trivial normal bundle [BLPP08, Sch02].

Note that the harmonicity (5) of the conformal Gauss map of a Willmore surface can be read as a holomorphicity condition: decomposing the trivial connection $d = d_+ + d_-$ into S commuting and anti-commuting parts, the type decompositions are

$$d_+ = \partial + \bar{\partial}, \quad d_- = A + Q$$

with complex holomorphic and anti-holomorphic structures ∂ and $\bar{\partial}$ and Hopf fields A and Q . Then

$$K \text{End}_-(\mathbb{H}^2) = \{\omega \in \Omega^1(\text{End}(\mathbb{H}^2)) \mid *\omega = S\omega = -\omega S\}$$

can be equipped [BFL⁺02] with a complex holomorphic structure $\bar{\partial}$ by putting

$$(\bar{\partial}_X B)(Y)\phi = \bar{\partial}_X(B(Y)\phi) - B(\bar{\partial}_X Y)\phi - B(Y)\partial_X \phi$$

for $B \in \Gamma(K \text{End}_-(\mathbb{H}^2))$ where $\phi \in \Gamma(\mathbb{H}^2)$, $X, Y \in \Gamma(TM)$, and $\bar{\partial}_X Y = \frac{1}{2}([X, Y] + J[JX, Y])$ is the complex holomorphic structure on TM .

Since the Hopf field A of the conformal Gauss map of a Willmore surface is a section of $K \text{End}_-(\mathbb{H}^2)$, the harmonicity $d * A = 0$ can be read as the condition that A is a holomorphic section of $\Gamma(K \text{End}_-(\mathbb{H}^2))$ since

$$d * A(X, J_M X) = -2(\bar{\partial}_X A)(X).$$

In particular, if $A \neq 0$ the zeros of A are isolated and the kernel of A defines a line subbundle $L_1 = \ker A$ of the trivial \mathbb{H}^2 bundle. Similarly, the Hopf-field Q is an anti-holomorphic section of the bundle $\bar{K} \text{End}_-(\mathbb{H}^2) = \{\omega \in \Omega^1(\text{End}(\mathbb{H}^2)) \mid *\omega = -S\omega = \omega S\}$, and the image of Q defines a line subbundle $L_{-1} = \text{im } Q$ of \mathbb{H}^2 . The corresponding two maps $f_1, f_{-1} : M \rightarrow S^4$ are either constant or branched conformal immersions [BFL⁺02].

Definition 5. If $A \neq 0$ then $\ker A = L_1$ is called the *forward Bäcklund transform* of f whereas for $Q \neq 0$ we call $\text{im } Q = L_{-1}$ the *backward Bäcklund transform* of f .

Since A and Q are essentially (2) the $(1, 0)$ and $(0, 1)$ part of dS , the Bäcklund transformation is an analogue of the $\bar{\partial}$ and ∂ construction for harmonic maps into \mathbb{CP}^n . To obtain a sequence of Willmore surfaces, we need to ensure that the conformal Gauss map of a Bäcklund transform extends smoothly into the branch points of the Bäcklund transform. Indeed:

Theorem 6 ([LP08]). *The conformal Gauss map of a Bäcklund transform f_i , $i = 1, -1$, of a Willmore surface is a harmonic complex structure on \mathbb{H}^2 . In particular, f_i is a (branched) Willmore surface.*

Thus, applying this procedure inductively, we obtain a sequence f_i , $i \in \mathbb{Z}$, of Willmore surfaces which only breaks down if $A_i = 0$, $Q_i = 0$ or f_i is constant. If the sequence breaks down at some point i , then the sequence is in fact finite:

Lemma 7 ([LP08]). *The possible sequences for a Willmore surface $f : M \rightarrow S^4$ are of the following form:*

- (i) $\circ - \overset{f}{\bullet} - \circ$,
- (ii) $\circ - \overset{f}{\bullet} -)$, $(- \overset{f}{\bullet} - \circ$,
- (iii) $(- \overset{f}{\bullet} -)$
- (iv) $(- \overset{f}{\bullet} - \bullet -)$, $(- \bullet - \overset{f}{\bullet} -)$, or
- (v) $\dots - \bullet - \overset{f}{\bullet} - \bullet - \dots$

where \bullet indicates a (non-constant) Willmore surface, \circ a point, and $)$ and $($ indicate that A and Q respectively are zero.

The finite sequences can be classified: we first note that $SL_i = L_i$ by (3) for the Bäcklund transforms f_i , $i = 1, -1$. In particular, if one of the Bäcklund transforms is a constant point $f_i = \infty$, then the image of f under the stereographic projection across ∞ of $S^4 = \mathbb{R}^4 \cup \{\infty\}$ to \mathbb{R}^4 has mean curvature sphere congruence S with $\infty \in S(p)$ for all p . In other words, the mean curvature sphere congruence of the surface in \mathbb{R}^4 degenerates to a plane, and f gives a minimal surface in \mathbb{R}^4 under the stereographic projection. The minimal surface has planar ends since $f_i = \infty \in L_q$ for some $q \in M$. On the other hand, in the case when $A = 0$ then f is the twistor projection of holomorphic curves in \mathbb{CP}^3 [BFL⁺02]. Finally, if $Q = 0$ then f is the dual surface of such a twistor projection.

Lemma 8 ([LP08]). *If the Willmore sequence of a Willmore surface $f : M \rightarrow S^4$ is finite then one of the following holds:*

- (i) f is after stereographic projection a minimal surface in \mathbb{R}^4 with planar ends, or
- (ii) f comes from the twistor projection of a complex holomorphic curve in \mathbb{CP}^3 .

Since minimal surface are given by complex holomorphic functions via the Weierstrass representation, the previous lemma can be read as the statement that a Willmore surface with finite Willmore sequence is given by complex holomorphic data.

Lemma 9 ([LPP05]). *Let $f : M \rightarrow S^4$ be a Willmore surface which allows a dual Willmore surface, that is $AQ = 0$. Then the Willmore sequence of f is finite if f has normal bundle degree $|\deg \perp_f| > 4(g - 1)$.*

In particular, since Willmore spheres have dual surfaces, we recover the results of Bryant [Bry84], Montiel [Mon00] and Ejiri [Eji88] for Willmore spheres. More generally, we have:

Theorem 10 ([LPP05], [LP08]). *If $f : S^2 \rightarrow S^4$ is a Willmore sphere, or if $f : T^2 \rightarrow S^4$ is a Willmore torus with non-trivial normal bundle, then f is given by complex holomorphic*

data. More precisely, f is after stereographic projection a minimal surface in \mathbb{R}^4 with planar ends or comes from the twistor projection of a complex holomorphic curve in \mathbb{CP}^3 .

Proof. As in the case of harmonic maps into \mathbb{CP}^n the proof relies on an estimate on the energy of the harmonic map: If $f : M \rightarrow S^4$ is a Willmore surface with at least n Bäcklund transforms, then the Plücker relation for quaternionic holomorphic curves [FLPP01] and a telescoping argument as in [Wol88] give an estimate on the Willmore functional of f

$$\frac{1}{4\pi}W(f) \geq -4n(n+1)(g-1) - n \deg \perp_f$$

where g is the genus of M . In particular, in the case when $g = 0$ the right hand side

$$4n(n+1) - n \deg \perp_f$$

tends to ∞ as n goes to ∞ , contradicting the finiteness of the Willmore energy. If $g = 1$ then the leading term on the right hand side is $-n \deg \perp_f$ and since f has non-trivial normal bundle we may assume, by passing to the dual surface if necessary, that $\deg \perp_f < 0$, which again gives a contradiction. Thus, in both cases the Willmore sequence is finite. \square

In particular, lemma 9 and the previous theorem are evidence for the following

Conjecture 1. Let $f : M \rightarrow S^4$ be a Willmore surface of a Riemann surface M of genus g into the 4-sphere with $|\deg \perp_f| > 4(g-1)$. Then f is given by complex holomorphic data.

4. μ -DARBOUX TRANSFORMS OF THE CONFORMAL GAUSS MAP

In [BLPP08] the Darboux transformation on conformal immersions from a Riemann surface into the 4-sphere is introduced. In the case of constant mean curvature surfaces parallel sections of the associated family of flat connections give Darboux transforms, the so-called μ -Darboux transforms. These are classical Darboux transforms [Dar99] only if $\mu \in \mathbb{R}_* \cup S^1$, but all μ -Darboux transforms have constant mean curvature [CLP10]. In particular, the induced transformation on the Gauss map again preserves harmonicity. We extend this transformation on harmonic maps to the case when S is the conformal Gauss map of a Willmore surface and show that this is a transformation on Willmore surfaces.

As usual for harmonic maps, we can introduce a spectral parameter and characterise harmonic complex structures on $\underline{\mathbb{H}}^2$ by the flatness of a \mathbb{C}_* -family of flat connections on the trivial \mathbb{C}^4 bundle over M . Here, we equip \mathbb{H}^2 with the complex structure I which is given by right multiplication by i , and thus identify $(\mathbb{H}^2, I) = \mathbb{C}^4$ by $\mathbb{H} = \mathbb{C} + j\mathbb{C}$, $\mathbb{C} = \text{span}_{\mathbb{R}}\{1, i\}$.

Theorem 11. *A complex structure $S \in \Gamma(\text{End}(\underline{\mathbb{H}}^2))$ is harmonic if and only if the family of complex connections*

$$d^\lambda = d + (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}$$

on $M \times \mathbb{C}^4 = (\underline{\mathbb{H}}^2, I)$ is flat. Here

$$A^{(1,0)} = \frac{1}{2}(A - I * A), \quad A^{(0,1)} = \frac{1}{2}(A + I * A)$$

denote the $(1,0)$ and $(0,1)$ parts of A with respect to the complex structure I .

Proof. The proof is a standard calculation using $[I, S] = 0$ and $Q \wedge A = 0$ to obtain the curvature of d^λ as

$$(6) \quad R^\lambda = (d * A)S((\lambda - 1)\pi_E + (\lambda^{-1} - 1)\pi_{E^\perp})$$

where E and E^\perp denote the $\pm i$ eigenspaces of S respectively, and

$$\pi_E = \frac{1}{2}(1 - IS), \quad \pi_{E^\perp} = \frac{1}{2}(1 + IS)$$

the projections along the orthogonal splitting $\underline{\mathbb{C}}^4 = E \oplus E^\perp$. Since (6) holds for all $\lambda \in \mathbb{C}_*$ we see that $d * A = 0$ if and only if $R^\lambda = 0$ for all $\lambda \in \mathbb{C}_*$. \square

At this point we are only interested in local theory, and thus will assume from now on that M is simply connected. Moreover, since a Willmore surface whose conformal Gauss map has Hopf field $A = 0$ is a twistor projection of a holomorphic curve in \mathbb{CP}^3 , we are primarily interested in harmonic complex structures with Hopf field $A \neq 0$. However, the above family of flat connections is gauge equivalent to a family of connections which are defined in terms of the Hopf field Q , so similar arguments as in the following could be used to deal with the case $A = 0$, $Q \neq 0$. If both Hopf fields vanish then S is constant.

Theorem 12. *Let S be a harmonic complex structure on $\underline{\mathbb{H}}^2 = M \times \mathbb{H}^2$ with $A \neq 0$ and d^λ the associated \mathbb{C}_* family of complex connections on $(\underline{\mathbb{H}}^2, I)$. For fixed $\mu \in \mathbb{C}_*$ let $\psi_1, \psi_2 \in \Gamma(\underline{\mathbb{H}}^2)$ be parallel sections of d^μ such that $W_\mu = \text{span}_{\mathbb{C}}\{\psi_1, \psi_2\}$ is a complex rank 2 bundle over M with $W_\mu \cap W_\mu j = \{0\}$. Then*

$$(7) \quad T = S(a - 1) + b$$

is invertible on M for $\mu \in \mathbb{C}_*$, $\mu \neq 1$, where $a = G \left(\frac{\mu + \mu^{-1}}{2} E_2 \right) G^{-1}$, $b = G \left(I \frac{\mu^{-1} - \mu}{2} \right) G^{-1}$ with identity matrix $E_2 \in \text{GL}(2, \mathbb{H})$ and $G = (\psi_1, \psi_2) \in \Gamma(\text{GL}(2, \mathbb{H}))$. Moreover,

$$\hat{S} = T^{-1} S T$$

is harmonic with Hopf fields

$$(8) \quad * \hat{A} = \frac{1}{2} T (1 - a)^{-1} * A T, \quad * \hat{Q} = 2 T^{-1} * Q (a - 1) T^{-1}.$$

We call \hat{S} the μ -Darboux-transform of the harmonic complex structure S . Note that \hat{S} is independent of the choice of basis of W_μ .

Proof. By assumption $\underline{\mathbb{C}}^4 = W_\mu \oplus W_\mu j$ so that every $\phi \in \underline{\mathbb{C}}^4$ has a unique decomposition $\phi = \phi_1 + \phi_2 j$ with $\phi_l \in W_\mu$. Decomposing $\phi_l = \phi_l^+ + \phi_l^-$ further according to the splitting $\underline{\mathbb{C}}^4 = E \oplus E j$ with $\phi_l^+ \in E, \phi_l^- \in E^\perp$ for $l = 1, 2$, we have a unique decomposition

$$\phi = \phi_1^+ + \phi_1^- + (\phi_2^+ + \phi_2^-) j.$$

Applying (7) we get

$$T \phi = \phi_1^+ i (\mu^{-1} - 1) + \phi_1^- i (1 - \mu) + \phi_2^+ j i (1 - \bar{\mu}) + \phi_2^- j i (\bar{\mu}^{-1} - 1),$$

and $T \phi = 0$ implies $\phi_1^+ i (\mu^{-1} - 1) = \phi_1^- i (1 - \mu) = \phi_2^+ j i (1 - \bar{\mu}) = \phi_2^- j i (\bar{\mu}^{-1} - 1) = 0$ because $\underline{\mathbb{C}}^4 = W_\mu \oplus W_\mu j = E \oplus E j$. Since $\mu \neq 1$ this shows that $\phi = 0$, and thus T is invertible.

The remainder of the proof is the exact analogue for the corresponding statement for harmonic complex structures J on $\underline{\mathbb{H}}$, see [CLP10]: Since $\underline{\mathbb{H}}^2 = W_\mu \oplus W_\mu j$ and W_μ is d_μ -parallel, the quaternionic extension of $d_\mu|_{W_\mu}$ is the quaternionic connection

$$\hat{d}^\mu = d + * A T$$

on $\underline{\mathbb{H}}^2$. Moreover, a, b are constant on the basis $\{\psi_1, \psi_2\}$ of W_μ so that $\hat{d}_\mu a = \hat{d}_\mu b = 0$, or expressed differently,

$$(9) \quad d(a - 1) = -[* A T, a - 1], \quad db = -[* A T, b].$$

Differentiating (7) we obtain the Riccati type equation

$$(10) \quad dT = *Q(a-1) + 2T * AT,$$

where we used (2) and $a^2 + b^2 = E_2$. Therefore, the derivative $d\hat{S} = [\hat{S}, T^{-1}dT] + T^{-1}dST$ of \hat{S} computes to

$$d\hat{S} = -2T^{-1} * Q((\hat{a}-1)\hat{S} + S(\hat{a}-1)) + 2(S + \hat{S}) * AT + T^{-1}dST$$

and the Hopf fields to

$$-2 * \hat{A} = (S + \hat{S} - 2T^{-1}) * AT, \quad * \hat{Q} = T^{-1} * Q(-(a-1)\hat{S} - S(a-1) + T).$$

Now $-2(a-1) + ST(a-1) - Tb = 0$ by (7), and thus

$$(11) \quad \hat{S} = 2T^{-1} + \frac{b}{a-1}$$

give the equations (8). Finally, (10) shows $dT^{-1} = -2T^{-1} * Q(a-1)T^{-1} - *A$, and

$$* \hat{Q} = -dT^{-1} - *A$$

is closed since S is harmonic. In other words, \hat{S} is harmonic by (5). \square

Remark 13. For $\mu \in S^1$ the μ -Darboux transform is trivial: in this case a and b are real multiples of the identity, and therefore $\hat{S} = S$ since $[T, S] = 0$.

The μ -Darboux transformation preserves the Willmore property:

Theorem 14. Let $f : M \rightarrow S^4$ be a Willmore surface which is not a twistor projection of a holomorphic curve in \mathbb{CP}^3 . Let S be the conformal Gauss map of f , then the μ -Darboux transform \hat{S} of S is the conformal Gauss map of a Willmore surface \hat{f} .

Proof. Let $\hat{S} = TST^{-1}$ be a μ -Darboux transform of S where T is defined (7) for two parallel sections of d^μ satisfying the assumptions of Theorem 12. We show that \hat{S} is the conformal Gauss map of

$$\hat{L} = T(a-1)^{-1}L.$$

With $a^2 + b^2 = E_2$ and (11) we obtain $\hat{S}T(a-1)^{-1} = T(a-1)^{-1}S$ and thus \hat{L} is \hat{S} stable since $SL = L$. Furthermore, by (8)

$$(12) \quad \text{im } \hat{A} \subset \hat{L} \subset \ker \hat{Q}$$

since $\text{im } A \subset L \subset \ker Q$. In particular, $d\hat{S} = 2(*\hat{Q} - *\hat{A})$ stabilises \hat{L} which gives $\hat{S}\hat{\delta} = \hat{\delta}\hat{S}$. Finally, for $\hat{\psi} \in \Gamma(\hat{L})$ we see

$$\hat{\delta} \wedge *\hat{A}\hat{\psi} = \pi_{\hat{L}} d \wedge (*\hat{A}\hat{\psi}) = 0$$

since \hat{S} is harmonic and $\text{im } \hat{A} \subset \hat{L}$. Since $\hat{A} \neq 0$ by (8) this shows that $\hat{\delta}$ has type $*\hat{\delta} = \hat{\delta}\hat{S}$. \square

Remark 15. The Darboux transformation as defined in [BFL⁺02] is the special case $\mu \in S^1 \cup \mathbb{R}_*$, $\mu \neq 1$ of our μ -Darboux transformation. In this case, the map a is a real multiple of the identity, and the Riccati type equation (10) becomes the Riccati equation as in [BFL⁺02] and (11) becomes the corresponding initial condition.

Remark 16. A similar theorem holds for constrained Willmore surfaces [Les]: an immersion $f : M \rightarrow S^4$ is constrained Willmore [BPP08] if $d(*A + \eta) = 0$ where S is the conformal Gauss map of the constrained Willmore immersion $f : M \rightarrow S^4$ and $\eta \in \Omega^1(\text{Hom}(\mathbb{H}^2/L, L))$ is the potential of f with $*\eta = S\eta = \eta S$. The complex structure S then gives rise to a family of flat connections $d^\lambda = d + (*A + \eta)(S(a-1) + b) - 2*\eta(a-1)$

with $a = E_2 \frac{\lambda + \lambda^{-1}}{2}, b = I \frac{\lambda^{-1} - \lambda}{2}, \lambda \in \mathbb{C}_*$. Given two parallel section of d_λ we define T by (7) and the complex structure by $\hat{S} = T^{-1}ST$. Then \hat{S} is the conformal Gauss map of a constrained Willmore surface with potential $\hat{\eta} = T(a - 1)^{-1}\eta(a - 1)T^{-1}$.

In [BDLQ] it is shown that the μ -Darboux transform of the Gauss map of a constant mean curvature surface is given by a simple factor dressing. In [Qui08] a dressing transformation on Willmore surfaces, and more generally on constrained Willmore surfaces, is introduced. Moreover, Quintino shows that the Darboux transforms as defined in [BFL⁺02], that is by Remark 15 the μ -Darboux transforms of the conformal Gauss map with $\mu \in \mathbb{R}_* \cup S^1$, are given by a simple factor dressing. We expect [Les] that the computations in [BDLQ] transfer to the Willmore case, and that a simple factor dressing of a harmonic complex structure is indeed a μ -Darboux transform for $\mu \in \mathbb{C}_*$.

In [BLPP08] a Darboux transform on conformal tori is defined which, even in the case of a Willmore surface, differs from the Darboux transform on the conformal Gauss map in Theorem 14: Recall that for a conformal immersion $f : M \rightarrow S^4$ a section $\psi^\sharp \in \Gamma(L^\sharp)$ defines a *Darboux transform* $L^\sharp = \psi^\sharp \mathbb{H}$ of f if $d\psi \in \Omega^1(L)$ where L is the line bundle of f . In particular, if f is Willmore and ψ_1, ψ_2 are parallel sections of d^μ for fixed $\mu \in \mathbb{C}_*$ as in Theorem 12, we have

$$(13) \quad d\psi_l = - * AT\psi_l \in \Omega^1(L), \quad l = 1, 2,$$

since $\text{im } A \subset L$. Thus $L^l = \psi_l \mathbb{H}$ are Darboux transforms of f . For $\mu \in S^1$ we see with $[T, S] = 0$ that $*d\psi_l = *ATS\psi_l$, that is $*\delta_l\psi_l = -\delta_i S\psi_l$. In particular, see Remark 13, the μ -Darboux transform $\hat{S} = S$ of S is not the conformal Gauss map of L^l .

Since the μ -Darboux transformation is defined by two parallel sections and gives a change of sign on the complex structure in the case when $\mu \in S^1$, we rather expect to see that a μ -Darboux transform is the conformal Gauss map of a two-fold Darboux transform of the backward Bäcklund transform; we will return to this topic in a future paper.

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